NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1244

ROTATIONALLY SYMMETRIC POTENTIAL FLOWS

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NACA

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1. CHARACTERISTIC DIFFERENTIAL EQUATIONS

The differential equations for the rotationally symmetric potential flows read

$$\left(1 - \frac{u^2}{a^2}\right)\varphi_{zz} - 2\frac{uv}{a^2}\varphi_{rz} + \left(1 - \frac{v^2}{a^2}\right)\varphi_{rr} + \frac{\varphi_r}{r} = 0 \tag{1}$$

if one starts from the velocity potential φ, and

$$\left(1 - \frac{u^2}{a^2}\right)\psi_{zz} - 2\frac{uv}{a^2}\psi_{rz} + \left(1 - \frac{v^2}{a^2}\right)\psi_{rr} - \frac{\psi_r}{r} = 0 \tag{2}$$

if one uses the stream function ψ (see chapter I, equations (21) and (19)). Therein the velocity component in z direction is

$$u = \varphi_z = (1 - q^2)^{-\frac{1}{\kappa-1}} \frac{1}{r}\psi_r \tag{3a}$$

the velocity component in r-direction

$$v = \varphi_r = -(1 - q^2)^{-\frac{1}{\kappa-1}} \frac{1}{r}\psi_z \tag{3b}$$

while for the sonic velocity a in the selected non-dimensional
representation

$$a^2 = \frac{\kappa - 1}{2}(1 - q^2) \tag{4}$$

applies. q was the amount of the velocity, thus $\sqrt{u^2 + v^2}$.

In setting up the characteristic differential equations one starts
best from (1), since the derivation then is somewhat facilitated. Later
on, it is true, interest will be concentrated on the stream
lines ψ = const., not on the potential lines φ = const; however, for
potential flows the quantities φ or ψ enter only into the last
characteristic differential equations whereas those treated first are
completely free of φ or ψ and contain, aside from the independent
variables, only u and v. u and v are more simply expressed by φ
than by ψ.

The characteristic condition (cf. chapter II, equation (8), NACA
TM 1242) is

$$\left(1 - \frac{u^2}{a^2}\right)\dot{r}^2 + 2uv\dot{z}\dot{r} + \left(1 - \frac{v^2}{a^2}\right)\dot{z}^2 = 0 \tag{5}$$

If one now introduces the angle ϑ of the stream line and the symmetry
axis (z-axis) of the flow, one obtains

$$u = q \cos \vartheta \qquad v = q \sin \vartheta \tag{6}$$

Due to the rotational symmetry, ϑ may be assumed between 0 and $\kappa/2$,
if reverse flows are excluded.

Then the characteristic condition turns into

$$(a^2 - q^2\cos^2\vartheta)\dot{r}^2 + 2q^2\sin \vartheta \cos \vartheta\dot{z}\dot{r} + (a^2 - q^2\sin^2\vartheta)\dot{z}^2 = 0 \tag{7}$$

Two roots λ' and λ'' result from this relation for $\dfrac{dr}{dz}$, the slope of the characteristic base curves with respect to the z-axis:

$$\lambda' = \frac{q^2 \sin \vartheta \cos \vartheta + a\sqrt{q^2 - a^2}}{q^2 \cos^2 \vartheta - a^2} \tag{8a}$$

$$\lambda'' = \frac{q^2 \sin \vartheta \cos \vartheta - a\sqrt{q^2 - a^2}}{q^2 \cos^2 \vartheta - a^2} \tag{8b}$$

As differential equation for the first family of characteristic base curves there results

$$dr - \lambda' \, dz = 0 \tag{9a}$$

for the second family

$$dr - \lambda'' \, dz = 0 \tag{9b}$$

(see chapter II, equations (28a), (29a), NACA TM 1242).

For interpretation of these relations (9a) and (9b) - rather complicated due to the form of λ' and λ'' - it is best to start from the equivalent statement (7). For the latter one may write

$$q^2(\cos^2 \vartheta \; \dot{r}^2 - 2 \sin \vartheta \cos \vartheta \; \dot{z}\dot{r} + \sin^2 \vartheta \; \dot{z}^2) = a^2(\dot{r}^2 + \dot{z}^2)$$

or, with introduction of differentials,

$$\frac{(\cos \vartheta \; dr - \sin \vartheta \; dz)^2}{dr^2 + dz^2} = \frac{a^2}{q^2} \tag{10}$$

Then there is on the left side the square of the vector product of the unit vectors $(\cos \vartheta, \sin \vartheta)$ and $\left(\dfrac{dr}{\sqrt{dr^2 + dz^2}}, \dfrac{dz}{\sqrt{dr^2 + dz^2}} \right)$, which are tangent to the stream line or the characteristic base curve, respectively.

If one denotes the angle between stream line and characteristic base curve by α, (10) therefore states that

$$\sin^2 \alpha = \frac{a^2}{q^2} \qquad (11)$$

This, however, is nothing else but the well-known Mach relation: α is the Mach angle between stream lines and Mach waves which are accordingly recognized as being identical with the characteristic base curves.

Hence the interpretation of the equations (9a) and (9b) results directly. The positive z direction is assumed to be in the direction of the mean flow. Noting the signs of the roots in the expressions for λ' and λ'' one then finds that the first family of characteristic

base curves (9a) forms with the stream lines the Mach angle $\alpha = \arcsin \frac{a}{q}$ to the left, looking in flow direction; the second family of characteristic base curves forms the same angle to the right.
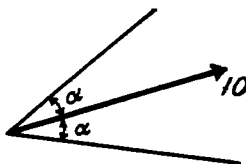
The first family of characteristic base curves in our specification may be designated as left-hand Mach waves, the second family as right-hand Mach waves. The second characteristic equation of the first family of characteristics in the form (chapter II, 28 b, NACA TM 1242) reads in our case:

$$\left(1 - \frac{u^2}{a^2}\right)\lambda' \, du + \left(1 - \frac{v^2}{a^2}\right)dv + \frac{v}{r} \, dr = 0 \qquad (12)$$

If one takes into consideration that in the expression (8a) for λ' the denominator is $q^2 \cos^2 \vartheta - a^2 = u^2 - a^2$, there results from (12) after multiplication by a^2 and consideration of (6):

$$-\left(q^2 \sin \vartheta \cos \vartheta + a\sqrt{q^2 - a^2}\right)\left(\cos \vartheta \, dq - \sin \vartheta q \, d\vartheta\right)$$

$$+\left(a^2 - q^2 \sin^2 \vartheta\right)\left(\sin \vartheta \, dq + \cos \vartheta q \, d\vartheta\right) + a^2 q \sin \vartheta \frac{dr}{r} = 0$$

or, after the multiplication of the brackets has been carried out and the terms have been rearranged,

$$\left\{\left(a^2 - q^2\right)\sin \vartheta - a\sqrt{q^2 - a^2} \cos \vartheta\right\}dq + \left\{a^2\cos \vartheta + a\sqrt{q^2 - a^2} \sin \vartheta\right\}q \, d\vartheta$$

$$+ a^2 q \sin \vartheta \frac{dr}{r} = 0$$

If one factors q from the first two terms, one obtains

$$q\left(a^2\cos \vartheta + a\sqrt{q^2 - a^2} \sin \vartheta\right)\left(d\vartheta - \frac{\sqrt{q^2 - a^2}}{aq} dq\right) + a^2 q \sin \vartheta \frac{dr}{r} = 0$$

or

$$\left(\cos \vartheta + \frac{\sqrt{q^2 - a^2}}{a} \sin \vartheta\right)\left(d\vartheta - \frac{\sqrt{q^2 - a^2}}{aq} dq\right) + \sin \vartheta \frac{dr}{r} = 0 \qquad (13)$$

If one now presupposes that

$$\cos \vartheta + \frac{\sqrt{q^2 - a^2}}{a} \sin \vartheta \neq 0 \qquad (14)$$

one may divide the equation (13) by this expression and obtain

$$d\vartheta - \frac{\sqrt{q^2 - a^2}}{aq} dq + \frac{1}{\cot \vartheta + \frac{\sqrt{q^2 - a^2}}{a}} \frac{dr}{r} = 0 \qquad (15)$$

Using the Mach angle, one may equate in (15)

$$\frac{\sqrt{q^2 - a^2}}{a} = \cot \alpha \qquad (16)$$

Accordingly, one has for the second family of characteristics

$$d\vartheta + \frac{\sqrt{q^2 - a^2}}{aq} \, dq + \frac{1}{\cot \vartheta - \frac{\sqrt{q^2 - a^2}}{a}} \, \frac{dr}{r} = 0 \tag{17}$$

with the presupposition

$$\cos \vartheta - \frac{\sqrt{q^2 - a^2}}{a} \sin \vartheta \neq 0 \tag{18}$$

What is the significance of the cases excluded by (14) and (18), respectively? If, for instance, (18) does not apply, one obtains $\vartheta = \alpha$. Since the stream line forms with the z-axis the angle ϑ, with the characteristic base curve the angle α, the respective characteristic base curve then becomes parallel to the z-axis; thus one has on it at this point $dr = 0$, the last term in (17) would assume the indefinite form $\infty \times 0$.

One may avoid these difficulties completely by using instead of the differential dr in (15) and (17) the differential ds, the arc element of the characteristic base curves. Using dz, which suggests itself would, in contrast, only shift the difficulty elsewhere.

If in the differential equation (15) pertaining to the first family of characteristics the factor $\frac{dr}{r}$ is designated by R', it is therefore to be equated

$$R' \, \frac{dr}{r} = S' \, \frac{ds}{r} \tag{19}$$

with precisely the expression S' to be determined. If one denotes the angle of the first family of characteristic base curves and the z-axis by β', there is $dr = \sin \beta' \, ds$. On the other hand, the slope of this characteristic base curve was given by λ' according to (8a), thus $\tan \beta' = \lambda'$. Therewith dr becomes $dr = \frac{\lambda'}{\sqrt{\lambda'^2 + 1}} \, ds$; thus, since there shall be $R' \, dr = S' \, ds$,

$$S' = \frac{\lambda' R'}{\sqrt{\lambda'^2 + 1}} \tag{20}$$

After a few transformations one obtains

$$\lambda'R' = -\frac{a\sin\vartheta}{a\sin\vartheta - \sqrt{q^2 - a^2}\cos\vartheta}$$

$$\sqrt{\lambda'^2 + 1} = q\,\frac{a\sin\vartheta + \sqrt{q^2 - a^2}\cos\vartheta}{q^2\cos^2\vartheta - a^2}$$

and finally

$$S' = \frac{a}{q}\sin\vartheta = \sin\alpha\sin\vartheta \tag{21}$$

Correspondingly

$$\lambda'R'' = -\frac{a\sin\vartheta}{a\sin\vartheta + \sqrt{q^2 - a^2}\cos\vartheta}$$

$$\sqrt{\lambda''^2 + 1} = -q\,\frac{a\sin\vartheta - \sqrt{q^2 - a^2}\cos\vartheta}{q^2\cos^2\vartheta - a^2}$$

The signs are selected so that, in agreement with the geometrical derivation above, $\dfrac{\lambda''}{\sqrt{\lambda''^2 + 1}} = \sin\beta''$ (β'' angle of the second family of characteristic base curves with the z-axis). Therewith S'' finally becomes

$$S'' = -\frac{a}{q}\sin\vartheta = -\sin\alpha\sin\vartheta \tag{22}$$

Thus one obtains without any distinctions of cases as second characteristic equation for the first family:

$$d\vartheta - \cot\alpha\,\frac{dq}{q} + \sin\vartheta\,\sin\alpha\,\frac{ds}{r} = 0 \tag{23}$$

and for the second family:

$$d\vartheta + \cot\alpha\,\frac{dq}{q} - \sin\vartheta\,\sin\alpha\,\frac{ds}{r} = 0 \tag{24}$$

Only this simplification of the characteristic differential equations
enabled us to perform the calculation of rotationally symmetric
potential flows with tolerable calculation expenditure. To our knowledge,
no indication of the possibility of such a simplification is to be found
in the suggestions of other authors (Frankl, Ferrari) concerning the
calculation of rotationally symmetric potential flows[1].

2. TREATMENT OF PRACTICAL EXAMPLES

For the calculation of practical examples, two methods are at
disposal for the approximate solution of the characteristic differential
equation system, as was discussed in more detail in chapter II, 7,
NACA TM 1242; we decided for the "field method" (chapter II, p. 16, NACA
TM 1242, p. 16). If one progresses from already known fields to new
ones, the equations (cf. chapter II, formula (49), p. 23, TM 1242, p. 24[2])

$$A_I \lambda_I{}' \left(p_{III} - p_I\right) + C_{II}\left(q_{III} - q_I\right) + D_I\left(y_{III} - y_I\right) = 0 \qquad (25a)$$

$$A_{II} \lambda_{II}{}'' \left(p_{III} - p_{II}\right) + C_{II}\left(q_{III} - q_{II}\right) + D_{II}\left(y_{III} - y_{II}\right) = 0 \qquad (25b)$$

are used, which result from the following characteristic differential
equations (II, (28), (29), p. 14, TM 1242, p. 15):

$$A\lambda{}' \, dp + C \, dq + D \, dy = 0 \qquad (26a)$$

$$A\lambda{}'' \, dp + C \, dq + D \, dy = 0 \qquad (26b)$$

with λ' signifying the slope of the first family of characteristic base
curves, λ'' signifying the slope of the second family of characteristic
base curves; the differentials are approximately replaced by finite
differences (and differences of second and higher order then neglected)

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[1]As noted afterwards, for once the arc element of the character-
istic base curves has been introduced into the characteristic differ-
ential equations from a purely mathematical side: (E. Kamke's
"Differentialgleichungen reeller Funktionen," (Differential equations of
real functions), Leipzig 1930, p. 389 (9)). We reserve for a later time
looking into a possible connection of Kamke's expressions of the
characteristic differential equations with ours.

[2]From now on, these references will be abbreviated; for this
reference, for instance, we shall write: II (49), p. 23, TM 1242, p. 24.

as has been described in detail before (II, p. 16 ff. and p. 22 ff. etc.,
TM 1242 p. 16 ff. and p. 23 ff.) I and II signify the numbers of the
fields with already known approximate values of p and q whereas III
denotes the adjoining field for which these approximate values are to
be determined.

The relations (25), derived from the quite general form of a
quasilinear differential equation of the second order (II, 1(1), p. 2,
TM 1242, p. 1) are now to be given for a special use in adjustment to
the flow type of importance.

For the <u>rotationally symmetric potential flow</u> there correspond to
the two equation (26) according to (23) and (24) the pair of equations
(second characteristic equation for the first and for the second family
of base curves)

$$d\vartheta - \cot \alpha \, \frac{dq}{q} + \sin \vartheta \, \sin \alpha \, \frac{ds}{r} = 0 \qquad (27a)$$

$$d\vartheta + \cot \alpha \, \frac{dq}{q} - \sin \vartheta \, \sin \alpha \, \frac{ds}{r} = 0 \qquad (27b)$$

in addition, the Mach relation (first characteristic equation)

$$\sin \alpha = \frac{a}{q} \qquad (28)$$

is used. Therein a was the local sonic velocity, q the amount of
velocity, ϑ the angle of the stream line and the symmetry axis (z-axis),
α the Mach angle between stream lines and characteristic base curves[3],
r the distance of a point from the z-axis.

Within the scope of our approximation principle we write for (27a)
and (27b) in analogy to (25a) and (25b) - putting for abbreviation

$$\frac{\cot \alpha_\nu}{q_\nu} = \frac{\sqrt{q_\nu^2 - a_\nu^2}}{a_\nu q_\nu} = Q_\nu \qquad (29)$$

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[3]Cf. remark on p. 4

and

$$\sin \vartheta_\nu \sin \alpha_\nu = S_\nu \qquad (\nu = I, II) \qquad (30)$$

$$\vartheta_{III} - \vartheta_I - Q_I\left(q_{III} - q_I\right) + S_I \frac{s_{III} - s_I}{r_I} = 0 \qquad (31a)$$

(progressing along a left-hand family with the angle $\vartheta + \alpha$ toward the z axis) and

$$\vartheta_{III} - \vartheta_{II} + Q_{II}\left(q_{III} - q_{II}\right) - S_{II} \frac{s_{III} - s_{II}}{r_{II}} = 0 \qquad (31b)$$

respectively (progressing along a right-hand family with the angle $\vartheta - \alpha$ toward the z-axis); $s_{III} - s_\nu$ simply signifies the distance of the field centers III and ν from each other.

Following the formulas required are given; they are to be used in connection with the schemes given by figure 1.

From the two linear equations (31) q_{III} (and therewith, according to (28), also α_{III}) and ϑ_{III} for the unknown field III are easily found. The elimination of ϑ_{III} from (31a) and (31b) first yields

$$q_{III} = \frac{Q_I q_I + Q_{II} q_{II} + S_I \dfrac{s_{III} - s_I}{r_I} + S_{II} \dfrac{s_{III} - s_{II}}{r_{II}} + \vartheta_{II} - \vartheta_I}{Q_I + Q_{II}} \qquad (32)$$

by insertion of q_{III} into (31a) or (31b) then results

$$\vartheta_{III} = \vartheta_I + Q_I\left(q_{III} - q_I\right) - S_I \frac{s_{III} - s_I}{r_I} \qquad (33a)$$

or

$$\vartheta_{III} = \vartheta_{II} - Q_{II}\left(q_{III} - q_{II}\right) + S_{II} \frac{s_{III} - s_{II}}{r_{II}} \qquad (33b)$$

Of the two formulas (33) one is, of course, superfluous; however, it may be useful for control of the calculation.

The relations (32) and (33) are applied in the standard case
(cf. fig. 1a), where one deals with inner fields only; if, on the other
hand, one reaches, in progressing, the fixed boundary (case of fig. 1b
and 1c, respectively), where ϑ is prescribed by the boundary condi-
tion 4), one has, according to (31), the simpler formulas

$$q_{III} = \frac{Q_I q_I + S_I \dfrac{s_{III} - s_I}{r_I} + \vartheta_{III} - \vartheta_I}{Q_I} \qquad \text{(case 1b)} \qquad \text{(34a)}$$

and

$$q_{III} = \frac{Q_{II} q_{II} + S_{II} \dfrac{s_{III} - s_{II}}{r_{II}} + \vartheta_{II} - \vartheta_{III}}{Q_{II}} \qquad \text{(case 1c)} \qquad \text{(34b)}$$

respectively.

Conditions at the boundary of a free jet are quite analogous.
Only ϑ is now not prescribed but, on the contrary, to be determined
whereas pressure and therewith the amount of velocity q are known.
The solution of (31a) and (31b) with respect to ϑ yields[5], for given q

$$\vartheta_{III} = \vartheta_I + Q_I\left(q_{III} - q_I\right) - S_I \frac{s_{III} - s_I}{v_I} \qquad \text{(case 1b)} \qquad \text{(35a)}$$

and

$$\vartheta_{III} = \vartheta_{II} - Q_{II}\left(q_{III} - q_{II}\right) + S_{II} \frac{s_{III} - s_{II}}{v_{II}} \qquad \text{(case 1c)} \qquad \text{(35b)}$$

respectively

The formulas so far refer - as will generally be the case - to
progressing in flow direction. Now and then, however, (cf. our procedure
in the second example, section 4) it is also required to follow the flow
rearward and one then has the conditions sketched in figure 2.

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[4]If necessary, one assumes the boundary approximated by series of
straight lines $\vartheta = $ const.

[5]A peculiarity here experienced in the determination of the field
center will be treated in detail in the discussion of our second
example.

The corresponding formulas naturally follow directly from the former ones, one only has to replace the progressing from field I to field III by that from III' to II' and, in analogy, the progressing from II to III by that from III' to I'. Solution of the new equations with respect to $q_{III'}$ and $\vartheta_{III'}$ results[6] in

$$q_{III'} = \frac{Q_{I'}q_{I'} + Q_{II'}q_{II'} - S_{I'}\,\dfrac{s_{I'} - s_{III'}}{r_{I'}} - S_{II'}\,\dfrac{s_{II'} - s_{III'}}{r_{II'}} - \vartheta_{II'} + \vartheta_{I'}}{Q_{I'} + Q_{II'}}$$

(36)

$$\vartheta_{III'} = \vartheta_{I'} + Q_{I'}\left(q_{I'} - q_{III'}\right) - S_{I'}\,\frac{s_{I'} - s_{III'}}{r_{I'}} \qquad (37a)$$

and

$$\vartheta_{III'} = \vartheta_{II'} - Q_{II'}\left(q_{II'} - q_{III'}\right) + S_{II'}\,\frac{s_{II'} - s_{III'}}{r_{II'}} \qquad (37b)$$

respectively;

(case 2a);

$$q_{III'} = \frac{Q_{I'}q_{I'} - S_{I'}\,\dfrac{s_{I'} - s_{III'}}{r_{I'}} - \vartheta_{III'} + \vartheta_{I'}}{Q_{I'}} \quad ; \quad \vartheta_{III'} = \vartheta_{boundary}$$

(38a)

(case 2b);

$$q_{III'} = \frac{Q_{II'}q_{II'} - S_{II'}\,\dfrac{s_{II'} - s_{III'}}{r_{II'}} - \vartheta_{II'} + \vartheta_{III'}}{Q_{II'}} \qquad \vartheta_{III'} = \vartheta_{boundary}$$

(38b)

(case 2c).

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[6]For the free jet formulas for the progressing upstream were not required and therefore not derived.

Furthermore, knowledge of the following two additional remarks is important for the application of the formulas above.

1. For the rotationally symmetric flow it is sufficient to construct for a longitudinal section the flow pattern above the z-axis $\vartheta = 0$ (which may be interpreted as boundary), since the course below this axis is simply obtained by reflection on it. However, if one wants to calculate this course separately - for instance for purposes of control -, one must take into consideration that one is now dealing with a left-hand Cartesian z-r system. This is taken into account by providing in the formulas the quantities r_ν - in themselves, according to definition, positive - with the <u>negative</u> sign.

2. If one uses in the formulas a field center ν lying on the symmetry axis $\vartheta = 0$, there becomes for it $S_\nu = 0$ and simultaneously $r_\nu = 0$; the expression $S_\nu \dfrac{s_R - s_\vartheta}{r_\nu}$ then assumes an indefinite form. In this case the significant expression is simply to be <u>omitted</u>, since there results $\lim\limits_{r \to 0} \dfrac{S}{r} = 0$, for the following reasons: Along the stream line $\vartheta = 0$ there is $v = 0$ and, for reasons of symmetry, obviously also $\dfrac{dv}{dr} = 0$. Thus one has, since $v = q \sin \vartheta$:

$0 = \dfrac{dq}{dr} \sin \vartheta + q \cos \vartheta \dfrac{d\vartheta}{dr}$ and consequently $\dfrac{d\vartheta}{dr} = 0$ for $r = 0$. Now one may write for small values $\Delta\vartheta$ and Δr: $\dfrac{S}{r} = \dfrac{\sin \alpha \times \sin \vartheta}{r} \approx \sin \alpha \dfrac{\Delta\vartheta}{\Delta r}$. For transition to the limit $\Delta r \to 0$ there results, because of $\dfrac{d\vartheta}{dr} = 0$, also $\lim\limits_{r \to 0} \dfrac{S}{r} = 0$.

Moreover, the following has to be said in general for practical application of our method. It will be useful to represent, before starting the calculation proper, the values of Q and α as functions of q by means of the relations (4) $a^2 = \dfrac{\kappa - 1}{2}\left(1 - q^2\right)$ and (29) for the interval $a^2 = \sqrt{\dfrac{\kappa - 1}{\kappa + 1}} \leq q \leq 1$; of course, this has to be done for every α value separately. For this purpose a table was set up which indicates the Q values and α values for q steps of 0.02; for the intermediate values linear interpolation which is easily performed is sufficient. It was shown for the used κ-values ($\kappa = 1.405$; $\kappa = 1.18$) that for the chiefly important q domain (if q does not approach the unit too closely) Q may be practically regarded as linear function of q. The two tables for $\kappa = 1.405$ and $\kappa = 1.18$ on which the calculation of our examples is based are added at the end (pp. 26 and 27).

The quantity S, on the other hand, is determined from case to case. Its calculation may be performed very simply by slide rule according to (30), all the more so since the factor $\sin \alpha = \dfrac{a}{q}$ which appears as intermediate quantity in the calculation required for the setting up of the table may be taken directly from that table.

For the rest, it will be noted in using our formulas that certain combinations of quantities occur repeatedly, which means a saving in calculation expenditure.

The calculation and construction of the field III - which for us is to be representative for all fields newly to be determined (cf. fig. 1) - is performed as follows: first, q_{III} and ϑ_{III} are determined according to the respective formulas, with the quantities denoting length $s_{III} - s_\nu$ and r_ν to be taken from the drawing (its scale is of no importance since only the quotient of those lengths enters into the calculation). The table yields the pertaining values of Q_{III} and α_{III}, and with the aid of ϑ_{III} one can find S_{III}. (It has to be noted that Q and S in themselves are auxiliary calculation quantities which are required only for the continuation of the procedure.) There follows a geometrical process, the closing of field III which requires knowledge of the Mach angle α_{III} and of the stream line angle ϑ_{III}. At the corner point 1 (cf. fig. 1a) the angle $\vartheta_{III} + \alpha_{III}$, at the corner point 2 the angle $\vartheta_{III} - \alpha_{III}$ is subtended by the horizontal ($||$ z-axis) and the legs of both angles are made to intersect at the desired corner point 3. Since thereby, naturally, a certain moment of inaccuracy (aside from the errors caused by the approximation principle) enters, the position of the corner point 3 was occasionally checked analytically by means of the equations (II (42), p. 18, TM 1242, p. 20)

$$\left.\begin{array}{l} r_3 - r_1 = \left(z_3 - z_1\right) \times \tan\left(\vartheta_{III} + \alpha_{III}\right) \\[2mm] r_3 - r_2 = \left(z_3 - z_2\right) \times \tan\left(\vartheta_{III} - \alpha_{III}\right) \end{array}\right\} \qquad (39)$$

or, solved with respect to the coordinates z_s, r_s:

$$\left.\begin{array}{l} z_3 = \dfrac{z_1 \times \tan\left(\vartheta_{III} + \alpha_{III}\right) - z_2 \tan\left(\vartheta_{III} - \alpha_{III}\right) - r_1 + r_2}{\tan\left(\vartheta_{III} + \alpha_{III}\right) - \tan\left(\vartheta_{III} - \alpha_{III}\right)} \\[4mm] r_3 = r_1 + \left(z_3 - z_1\right) \times \tan\left(\vartheta_{III} + \alpha_{III}\right) \end{array}\right\} \quad (39')$$

After this general representation of our combined calculative-graphical method we now turn to the description of the special examples that have been carried out.

3. FIRST EXAMPLE: DIFFUSER

This example which we selected ourselves and which represents the first attempt of application of our method, deals with the flow of air ($\kappa = 1.405$) through a conical nozzle of the half opening angle 6° according to figure 3, which shows a longitudinal section. As initial condition we assumed q in this longitudinal section on a circular arc (the apex of the cone is the center of the circle), to be as constant throughout, equal 0.59, (the pertaining Mach angle is $38^\circ 2'$); ϑ results from the fact that the stream lines must intersect this circular arc radially. The distribution of the fields is here selected still relatively roughly (ϑ step of about 4° in r direction): field corners on the circular arc at $\vartheta = 0$ and $\vartheta = \pm 4^\circ$, thus field centers at $\vartheta = \pm 2^\circ$ and $\vartheta = \pm 6^\circ$ (boundary), as can be seen from figure 3. Also, the method was extended only to the construction of a few fields since no peculiarities appear.

The construction in general has already been described in detail; supplementary information is needed only about the determination of the field centers. For inner fields one finds the new field center III (cf. fig. 1a) as center of the diagonal $\overline{12}$ of the corners 1 and 2 of the known fields I and II. At the boundary the procedure is as follows (cf. fig. 1b and, analogously, fig. 1c): the parallel to the characteristic base curve $\overline{12}$ of the field I is made to intersect with the boundary line, and the intersection point is defined as field "center" of the half-field III.

The ratio of the prevailing pressure p to the tank pressure p_0 (that is, the pressure for which the flow velocity equals zero) has been plotted into the individual fields. This ratio results, according to I, 2, from the respective q according to the equation

$$\frac{p}{p_0} = (1 - q^2)^{\frac{\kappa}{\kappa-1}} \tag{40}$$

Furthermore, a number of stream lines are drawn; they were selected according to the principle that in every stream tube the same mass transport takes place per unit time. These stream lines are attained by progressing by equal amounts $\Delta\psi$, starting from $\vartheta = 0$. Their position is fixed in the following manner.

The point which was taken in figure 3 as starting point of the stream line $\vartheta = 0$ is assumed to be the origin of our z-r coordinate system. One now searches along $z = 0$ for the r ordinate of the points through which the stream lines must pass. For constant z one has, if the first equation (16) of chapter I, p. 9 is written as difference equation,

$$\Delta\psi \approx r\ q\ \cos\ \vartheta(1 - q^2)^{\frac{1}{\kappa-1}}\ \Delta r$$

$$\approx P\ \Delta r \tag{41}$$

According to the Simpson rule[7]

$$\psi_{boundary} - \psi_0 = \int_{r=0}^{r=r_{boundary}} P\ dr \tag{42}$$

was numerically determined (which requires setting up of a table for a number of r values and of the pertaining values of P) and

$$\Delta\psi = \frac{\psi_{boundary} - \psi_0}{n} \tag{43}$$

was then found, with n signifying the number of stream lines to be plotted (in the present case $n = 6$). Insertion of $\Delta\psi$ in (41) then yields the n different Δr by which one must progress along $z = 0$, starting from $r = 0$. It is practical to proceed from the boundary: one introduces $P(r = r_{boundary})$ into (41) and thus obtains $\Delta_1 r$. Then one determines by interpolation from the table the P value pertaining to $r_{boundary} - \Delta_1 r$ and inserts it again in (41), thus obtaining $\Delta_2 r$. The same procedure is performed for $P = P(r_{boundary} - \Delta_1 r - \Delta_2 r)$ and yields $\Delta_3 r$, etc. Naturally one must not expect $\sum_{\nu=1}^{n} \Delta_\vartheta r$ to become now exactly equal to $r_{boundary}$, since first, always a too large P was used and, second, P was regarded as constant for the entire (finite) interval Δr. Within the scope of graphical accuracy, however, it proved sufficient to plot now r against $\Delta\psi$, to alter the curve drawn through the r_i ($i = 1, 2, \ldots, n$) so that it passes exactly through $r = 0$ for $n\ \Delta\psi = \psi_{boundary} - \psi_0$ (insignificant correction!), and to use the new coordination from the $r-\Delta\psi$ diagram. Besides, one may also, as a proof, proceed inversely starting from $r = 0$; one then obtains too large values for Δr, whereas those found according to the first method were too small.

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[7]Of course, another suitable quadrature formula could have been selected just as well.

The stream lines start at the angle ϑ given by the initial condition (the slight difference between the circular arc and the rectilinear section $z = 0$ could be neglected); when they reach the field boundary, they are continued at the new inclination coordinated to the adjoining field and thus result in the broken lines shown in figure 3.

4. SECOND EXAMPLE: NOZZLE

Here the flow of a gas with $\kappa = 1.18$ through a nozze of prescribed longitudinal dimensions, the shape of which in longitudinal section is given in figure 4, was to be determined. The nozzle is a truncated circular cone with the wall angle $6°10'$. The narrowest cross section has a diameter of 118 millimeters, the exit cross section, one of 178 millimeters. At the narrowest cross section the obtuse cone starts with a slight - not further defined - curvature. The location of transition is marked in our drawing. The tank pressure is 14 atmospheres, the pressure in the exit cross section 1 atmosphere. The determination of the curve results necessarily in our procedure which conforms to a report of Th. Meyer[8] and will be described in detail below.

In the surroundings of the narrowest nozzle cross section where the flow velocity is near critical velocity, our method is not yet usable in practice (and this applies to the supersonic region, also), because the Mach angle α still differs too little from $90°$ and the fields thus become much too narrow to allow even a tolerably rapid progress in the construction of the flow pattern. For this reason we apply our method only at a sufficient distance from the narrowest cross section (naturally making our arrangements so that we need operate only with the rectilinear boundary); it is true that initial conditions for q and ϑ for the start then must be obtained by another method.

This latter, according to Th. Meyer, consists in assuming at the point of the z axis where sonic velocity is passed as origin of the coordinate system a series development with respect to the two coordinates z and r for the velocity potential φ. It is presupposed that in the proximity of the origin the first series terms already give a good approximation; proof of this will be given later (cf. p. 23). The coefficients of this development result by insertion of the series into the differential equation for φ (I (21), p. 12)

[8]Meyer, Th.: Über zweidimensionale Bewegungsvorgänge in einen Gas, das mit Überschallgeschwindigkeit strömt. Diss. Göttingen 1907, published - in Mitteilungen über Forsch.- Arb. Ing.-Wes. Heft 62, Berlin 1908.

and subsequent comparison of coefficients. However, certain coeffi-
cients remain arbitrary as long as no further conditions are added. The
first condition is obviously that the flow should be symmetrical to
the z axis, that is, v must not contain other than odd powers of r
or φ other than even powers of r. As second condition Meyer intro-
duces the assumption (well confirmed by tests) of a linear velocity
increase along the z axis (with the critical sonic velocity at the
origin).

A priori, another method of determining the coefficients would be
more readily apparent: it is the method used by G. I. Taylor. Taylor[9]
and with him S. G. Hooker[10] who transferred Taylor's method to the
three-dimensional case of a nozzle of circular cross section, do not
have Meyer's second condition but prescribed for their calculations a
certain nozzle with circular-arc boundary. The boundary conditions
then yielded the lacking qualifying equations for the coefficients.
However, it becomes evident that in this case the determination of the
coefficients becomes more complicated than according to Meyer's method,
quite aside from the fact that the latter method is better adapted to
our statement of the problem where the form of the boundary is not
accurately fixed. Moreover, the results of T. E. Stanton's[11] measure-
ments on the nozzle selected by Hooker confirms the usefulness of
Meyer's assumption (see above), that is, one obtains in this manner
reasonable nozzle shapes.

If one denotes all quantities made non-dimensional by a dash above
them and refers the velocity to the critical sonic velocity a*, Meyer's
second condition reads:

$$\bar{u} = 1 + \bar{z} \tag{44}$$

[9]Taylor, G. I.: The Flow of Air at High Speeds past Curved
Surfaces. Aeronautical Research Committee Reports and Memoranda
No. 1381, London 1930.

[10]Hooker, S. C.: The Flow of a Compressible Liquid in the Neigh-
bourhood of the Throat of a Constriction in a Circular Wind Channel.
Proc. roy. Soc. London A, Bd. 135 (1932), pp. 498-511.
Cf. also Görtler, H.: Zum Übergang von Untershall= zu Uber-
schallgeschwindigkeiten in Düsen. Zeitsch. f. angew. Math. Mech.
Bd. 19, Nr. 6, Lez. 1939.

[11]Stanton, T. E.: Velocity in a Wind Channel Throat. Aero-
nautical Research Reports and Memoranda No. 1388, London 1931.

The stream lines satisfy the relation

$$\frac{d\bar{r}}{d\bar{z}} = \frac{\bar{\varphi}\bar{r}}{\bar{\varphi}\bar{z}} = \frac{\bar{v}}{\bar{u}} \tag{45}$$

with $\bar{r}$ expressed as power series in $\bar{z}$, the coefficients of which are to be determined according to (45).

If one proceeds up to terms of the fourth order, the expression for the velocity potential becomes - with the first condition taken into consideration

$$\bar{\varphi} = a_1 z + a_2 \bar{z}^2 + c_2 \bar{r}^2 + a_3 \bar{z}^3 + b_3 \bar{z}\bar{r}^2 + a_4 \bar{z}^4 + b_4 \bar{z}^2 \bar{r}^2 + c_4 \bar{r}^4 + \cdots ;$$

$$\tag{46}$$

because of the second condition one has $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = a_4 = 0$, consequently (46) may be replaced by the simpler form

$$\bar{\varphi} = \bar{z} + \frac{1}{2}\bar{z}^2 + c_2 \bar{r}^2 + b_3 \bar{z}\bar{r}^2 + b_4 \bar{z}^2 \bar{r}^2 + c_4 \bar{r}^4 + \cdots \tag{47}$$

with the four unknown coefficients c_2, b_3, b_4, c_4. Hence results

$$\bar{u} = \frac{\partial\bar{\varphi}}{\partial\bar{z}} = 1 + \bar{z} + b_3 \bar{r}^2 + 2b_4 \bar{z}\bar{r}^2 + \cdots \tag{48a}$$

and

$$\bar{v} = \frac{\partial\bar{\varphi}}{\partial\bar{r}} = 2c_2 \bar{r} + 2b_3 \bar{z}\bar{r} + 2b_4 \bar{z}^2 \bar{r} + 4c_4 \bar{r}^3 + \cdots \tag{48b}$$

Insertion into the differential equation I (21)

$$\left(\bar{a}^2 - \bar{u}^2\right)\bar{\varphi}_{\bar{z}\bar{z}} - 2\bar{u}\bar{v}\bar{\varphi}_{\bar{r}\bar{z}} + \left(\bar{a}^2 - \bar{v}^2\right)\bar{\varphi}_{\bar{r}\bar{r}} + \frac{\bar{a}^2}{\bar{r}}\bar{\varphi}\bar{r} = 0$$

with

$$\bar{a}^2 = \frac{\kappa - 1}{2}\left(\frac{1}{a*^2} - \bar{u}^2 - \bar{v}^2\right)$$

leads by comparison of the coefficients to the four equations:

$$4c_2(\kappa - 1)\left(1 - a^{*2}\right) + \left[(\kappa - 1) - (\kappa + 1)a^{*2}\right] = 0 \qquad (49.1)$$

$$(\kappa - 1)\left\{4b_3 - 4a^{*2}b_3 - 8a^{*2}c_2\right\} - (\kappa + 1) \times 2a^{*2} = 0 \qquad (49.2)$$

$$(\kappa - 1)\left\{4b_4 - 4a^{*2}b_4 - 8a^{*2}b_3 - 4a^{*2}c_2\right\} - (\kappa + 1)a^{*2} = 0 \qquad (49.3)$$

$$(\kappa - 1)\left\{2b_4 + 16c_4 - 16a^{*2}c_4 - 4c_2{}^2 \right\} - \frac{\kappa}{a^{*2}} \times 16c_2{}^3 - (\kappa + 1)$$

$$\times \left(2a^{*2}b_4 + 2a^{*2}b_4 + 8c_2b_3\right) = 0 \qquad (49.4)$$

They yield for the unknown coefficients the values:

$$c_2 = 0 \qquad (50.1)$$

$$b_3 = \frac{1}{2\left(1 - a^{*2}\right)} \qquad (50.2)$$

$$b_4 = \frac{1 + 3a^{*2}}{4\left(1 - a^{*2}\right)^2} \qquad (50.3)$$

$$c_4 = \frac{1}{16\left(1 - a^{*2}\right)^2} \qquad (50.4)$$

Let the equation of the stream line up to terms of fourth order read :

$$\bar{r} = \alpha + \beta\bar{z} + \gamma\bar{z}^2 + \delta\bar{z}^3 + \epsilon\bar{z}^4 \qquad (51)$$

If one enters with (51) into (45), there follows according to (48):

$$\beta + 2\gamma\bar{z} + 3\delta\bar{z}^2 + 4\epsilon\bar{z}^3 = \frac{2c_2\bar{r} + 2b_3\bar{z}\bar{r} + 2b_4\bar{z}^2r + 4c_4\bar{r}^3}{1 + \bar{z} + b_3\bar{r}^2 + 2b_4\bar{z}\bar{r}^2} \qquad (52)$$

If one inserts, moreover, on the right side of (52) the value of $\bar{r}$ from (51), the comparison of the coefficients yields the following four relations[12]:

$$\beta(1 + b_3\alpha^2) = 4c_4\alpha^3 \tag{53.1}$$

$$2\gamma(1 + b_3\alpha^2) = 2b_3\alpha + (12c_4 - 2b_4)\alpha^2\beta - \beta - 2b_3\alpha\beta^2 \tag{53.2}$$

$$3\delta(1 + b_3\alpha^2) = 2b_3\beta + 2b_4\alpha - 2\gamma - b_3\beta^3 - 6b_3\alpha\beta\gamma - (4b_4 - 12c_4)\times(\alpha\beta^2 + \alpha^2\gamma) \tag{53.3}$$

$$4\epsilon(1 + b_3\alpha^2) = 2b_3\gamma + 2b_4\beta - 3\delta - 8b_3\alpha\beta\delta - 4b_3(\beta^2\gamma + \alpha\gamma^2)$$

$$- (12b_4 - 24c_4)\left(\alpha\beta\gamma + \frac{\beta^3}{6} + \frac{\alpha^2\delta}{6}\right) \tag{53.4}$$

The quantity α which indicates the $\bar{r}$ value (that is, half the nozzle width) at the point ($\bar{z} = 0$) where at the nozzle center the critical sonic velocity a* prevails, remains, as an integration constant, at first arbitrary. α will then serve for the determination of the scale which is used to make z and r non-dimensional.

We may now turn to the fixing of the curvilinear boundary section near the narrowest cross section of our nozzle (cf. p. 15) which had so far been left undetermined. This curve section must be stream line and must pass with continuous tangent into the rectilinear section with the prescribed inclination of 6°10'. Furthermore it must be observed that agreement with the prescribed dimensions is maintained and also that the z-interval is not exceeded (only within this z-interval the admissibility of our series expression may be regarded as satisfied). By trial an α value was determined which is coordinated to a stream line suitable for our purpose.

The performance of the numerical calculation yields for $\kappa = 1.18$ according to (50):

$$b_3 = 0.545 \qquad b_4 = 0.3705 \qquad c_4 = 0.0743$$

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[12]At this opportunity, a group of misprints in Meyer's report should be pointed out. On p. 62 one must read in his corresponding equations (always first term on the left side): in (2), 2γ instead of γ, in (3), 3δ instead of δ; in (4), 4ϵ instead of ϵ; in (5), 5ζ instead of ζ; in (6), 6η instead of η.

For α we selected the value 0.5 (it should be noted that this value must not be compared with Meyer's α = 0.5, Meyer having made his calculations for two-dimensional air flows with κ = 1.405, but that its corresponding value in Meyer's report - with respect to the course of the stream line - is probably about α = 0.2). Then (53) yields:

$$\beta = 0.03265 \qquad \gamma = 0.2258 \qquad \delta = -0.02677 \qquad \cdot \epsilon = 0.0651$$

The equation of the boundary section (= stream line) reads therefore, according to (51):

$$\bar{r} = 0.5 + 0.03265\bar{z} + 0.2258\bar{z}^2 - 0.02677\bar{z}^3 + 0.0651\bar{z}^4 \qquad (51^*)$$

From the condition

$$\frac{d\bar{r}}{d\bar{z}} = \tan 6°10' = 0.1081 = 0.03265 + 0.4516\bar{z} - 0.08031\bar{z}^2 + 0.2604\bar{z}^3$$

results for the $\bar{z}$ at the junction (in fig. 4 marked on the boundary)

$$\bar{z}_o = 0.1695$$

To this z_o pertains, according to (51*), the value

$$\bar{r}_o = 0.5119$$

α = 0.5 corresponds to the radius at the entrance of our nozzle r_o = 59 millimeters; thus we must multiply the above values of $\bar{z}$ and $\bar{r}$ by $\frac{59}{0.5}$ = 118, in order to find from Meyer's quantities the true lengths in millimeters for our example.

If one inserts the numerical values of the coefficients in (47) and (48), one obtains (a* = 0.287)

$$\bar{\varphi} = \frac{\varphi}{0.287} = \bar{z} + \frac{1}{2}\bar{z}^2 + 0.545\bar{z}\bar{r}^2 + 0.3705\bar{z}^2\bar{r}^2 + 0.0743\bar{r}^4 \qquad (47^*)$$

$$\bar{u} = \frac{u}{0.287} = 1 + \bar{z} + 0.545\bar{r}^2 + 0.7410\bar{z}\bar{r}^2 \qquad (48a^*)$$

$$\bar{v} = \frac{v}{0.287} = 1.090\bar{z}\bar{r} + 0.7410\bar{z}^2\bar{r} + 0.297\bar{r}^3 \qquad (48b^*)$$

With the aid of the equations (48*) one could finally, according to the relations

$$q = \sqrt{u^2 + v^2} \qquad \vartheta = \text{arc tan } \frac{v}{u}$$

calculate along the rectilinear section $\bar{z} = \bar{z}_0$ the initial values of q and ϑ which were requirements for our method.

The construction of the flow pattern in the interior of the nozzle to its exit proceeds in analogy to the construction of the first example, described before in detail; only the subdivision of the fields was now selected finer than in that example. Finally it should be mentioned that, starting from $z = z_0$, we constructed the flow pattern also for some length rearward and found the q values determined according to our method in good agreement with the corresponding values resulting from the series development.

Figure 5 shows the gas flow after leaving the nozzle for the case of an external pressure of 1 atmosphere. Figure 5 is the direct continuation of figure 4 and to be considered joined to it at the right. Since it could be seen that the subdivision of the fields in the nozzle interior was no longer fine enough for the construction of the free jet, it was made from new initial conditions for q and ϑ. These values were obtained by interpolation from the course of q and ϑ (discontinuous due to our approximation method) along the straight line $z = $ constant at the end of the nozzle.

The behavior of the jet depends on whether it enters, after leaving the nozzle, a region of higher or lower pressure. In our case the pressure at the boundary of the nozzle exit was found to be larger than the external pressure; it is true that the pressure difference proved to be only slight. Figure 5 indicates that $p_a/p_0 = 0.085$; since $p_0 = 14$ atmospheres, there follows $p_a = 1.2$ atmospheres. Thus there results as difference: 1.2 atmospheres - 1 atmosphere = 0.2 atmospheres. The boundary point forms the starting point of a rarefaction wedge with the opening angle $\alpha_1 - \alpha_a$ (with α_1 signifying the Mach angle pertaining to the internal pressure, α_a the Mach angle pertaining to the external pressure) which extends into the interior of the jet. Here this angle was so small (an ample half degree) that we could, within the scope of graphical accuracy, regard the rarefaction wedge as practically infinitely thin and coinciding with the characteristic base curve starting at that "singular" point. Still, one can see clearly in

figure 5 that the stream lines along this curve experience •a break exceeding the customary amount caused by the approximation[13].

Our construction deviates from the foregoing due to the fact that we deal no longer with a fixed boundary but with a free boundary to be determined, with the q_a pertaining according to (40) to the external pressure prescribed as boundary condition. First, the continuation of the straight boundary line ($\vartheta = 6°10'$) at the singular boundary point has to be determined, which is done in principle by formula (35a). However, both field centers I and III now coincide, precisely at the boundary point; consequently ds becomes zero, and the third term of the sum at right is eliminated. Since we may, as was mentioned before, assume Q in the interval in question $q_i \ldots q_a$ as linear function of q, ϑ_a may be found from $\vartheta_i = 6°10'$ without step-by-step difference calculation according to (35a) directly by an integration process:

$$\vartheta_a = \vartheta_i + \int_{q_i}^{q_a} Q \, dq \tag{52}$$

with $Q = \alpha q + \beta (\alpha, \beta$ constants which can easily be determined from the established table (cf. p. 26)).

If one wants to proceed further along the boundary, a new principle must be introduced for the geometrical determination of the field center on the boundary, since the one indicated for the fixed boundary now fails to work. Let the field I be known (cf. fig. 6). Point III is then fixed by elongating the line that so far formed the boundary (the first time the direction is given by ϑ_a calculated according to (52)) beyond the corner point I and having it intersect with the parallel to the section $\overline{23}$. One now lets the new direction ϑ_{III} found by calculation from (35a) start at III. The boundary stream line of the free jet therefore undergoes the (discontinuous) variation in its direction in the field center, in contrast to the inner stream lines where this variation takes place along the field boundaries.

We had actually intended to pursue the course of the jet until we had exhibited the periodicity in the expansion-contraction process. However, having progressed (on the outside) by about the length of the nozzle, we were forced to break off our procedure by an incident the

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[13]This deflection in itself takes place (in continuous transition) in the interior of the wedge and must be constructed (subdivision of $\alpha_i - \alpha_a$ into a number of $\Delta\alpha$-steps).

interpretation of which shall be postponed for the moment. It was found that two adjacent characteristics intersected (the point of intersection is outside of the range of fig. 5; however, the two curves in question are easily recognized). The problem arises whether we have to deal here only with an apparent singularity - caused by the inaccuracy of the approximation - or whether actually a compression shock occurs, although only a weak one. The latter is probable since we constructed the section of the free jet given by figure 5 twice and found in both cases an intersection of two adjacent characteristics; moreover, on the basis of our tests made in this direction one cannot reject the supposition that the same phenomenon will repeat itself at other points later.

As a conclusion it should be mentioned with regard to the second example that the ratio p/p_0 again, as in the first example, was written into the individual fields (cf. p. 15) and that the 24 stream lines have been selected according to the same principle.

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APPENDIX

TABLE I

SONIC VELOCITY a, $\frac{a}{q}$, AUXILIARY QUANTITY OF CALCULATION Q

AND MACH ANGLE α AS FUNCTIONS OF THE VELOCITY AMOUNT q

FOR $\kappa = 1.405$

| q	a	$\dfrac{a}{q}$	$Q = \dfrac{\sqrt{1 - \left(\dfrac{a}{q}\right)^2}}{a}$	$\alpha = \text{arc sin } \dfrac{a}{q}$
0.411	0.4128	1.000	0.000	90°
.43	.4062	.946	.806	71°5'
.45	.4019	.893	1.120	63°15'
.47	.3972	.845	1.344	57°37'
.49	.3920	.800	1.529	53°5'
.51	.3870	.758	1.682	49°17'
.53	.3813	.720	1.816	46°2'
.55	.3757	.683	1.946	43°5'
.57	.3693	.648	2.061	40°24'
.59	.3630	.616	2.172	38°2'
.61	.3562	.584	2.275	35°44'
.63	.3498	.556	2.379	33°47'
.65	.3415	.525	2.490	31°40'
.67	.3338	.4980	2.599	29°52'
.69	.3256	.4720	2.710	28°10'
.71	.3170	.4466	2.824	26°32'
.73	.3076	.4213	2.946	24°56'
.75	.2977	.3969	3.080	23°22'
.77	.2870	.3730	3.231	21°53'
.79	.2760	.3495	3.395	20°28'
.81	.2639	.3257	3.586	19°
.83	.2510	.3023	3.799	17°36'
.85	.2370	.2789	4.051	16°12'
.87	.2220	.2552	4.352	14°48'
.89	.2051	.2306	4.740	13°20'
.91	.1866	.2050	5.240	11°50'
.93	.1655	.1780	5.945	10°15'
.95	.1404	.1478	7.04	8°30'
.97	.1096	.1130	9.08	6°30'
.99	.0638	.0644	15.67	3°42'
1.00	.0000	.0000	∞	0°

TABLE II

SONIC VELOCITY a, $\frac{a}{q}$, AUXILIARY QUANTITY OF CALCULATION Q

AND MACH ANGLE α AS FUNCTIONS OF THE VELOCITY AMOUNT q

FOR κ = 1.18

q	a	$\frac{a}{q}$	$Q = \dfrac{\sqrt{1 - \dfrac{a}{q}^2}}{a}$	$\alpha = \text{arc sin } \dfrac{a}{q}$
0.287	0.2874	1.0000	0.000	90°
.30	.2862	.954	1.048	72°30'
.32	.2843	.889	1.610	62°40'
.34	.2821	.830	1.977	56°5'
.36	.2796	.777	2.251	51°
.38	.2775	.730	2.462	46°55'
.40	.2750	.688	2.640	43°25'
.42	.2724	.649	2.793	40°25'
.44	.2693	.612	2.936	37°42'
.46	.2663	.5795	3.058	35°28'
.48	.2633	.5490	3.172	33°17'
.50	.2598	.5195	3.288	31°20'
.52	.2563	.4932	3.397	29°33'
.54	.2530	.4686	3.491	27°56'
.56	.2486	.4440	3.608	26°21'
.58	.2445	.4218	3.710	24°57'
.60	.2400	.4000	3.819	23°34'
.62	.2354	.3800	3.931	22°20'
.64	.2304	.3602	4.050	21°12'
.66	.2254	.3417	4.168	19°58'
.68	.2200	.3236	4.300	18°53'
.70	.2142	.3061	4.447	17°50'
.72	.2081	.2891	4.601	16°49'
.74	.2017	.2722	4.770	15°48'
.76	.1949	.2563	4.961	14°52'
.78	.1877	.2406	5.17	13°55'
.80	.1800	.2250	5.41	13°
.82	.1717	.2093	5.70	12°5'
.84	.1627	.1938	6.02	11°10'
.86	.1531	.1781	6.42	10°15'
.88	.1425	.1620	6.92	9°20'
.90	.1308	.1454	7.56	8°22'
.92	.1176	.1278	8.43	7°20'
.94	.1023	.1088	9.72	6°15'
.96	.0841	.0876	11.84	4°50'
.98	.0597	.0609	16.74	3°26'
1.00	.0000	.0000	∞	0°

TABLE III

PRESSURE p REFERRED TO THE TANK

PRESSURE p₀ AS A FUNCTION OF

THE VELOCITY AMOUNT q

q	p/p_0 $\kappa = 1.405$	p/p_0 $\kappa = 1.18$
0.30		0.5388
.32		.4924
.34		.4467
.36		.4024
.38		.3597
.40		.3190
.42	0.5100	.2800
.44	.4741	.2440
.46	.4382	.2102
.48	.4030	.1796
.50	.3686	.1518
.52	.3350	.1265
.54	.3023	.1018
.56	.2710	.0849
.58	.2410	.0682
.60	.2127	.0536
.62	.1859	.0416
.64	.1608	.0316
.66	.1373	.0235
.68	.1161	.0171
.70	.0966	.0121
.72	.0791	.0083
.74	.0638	.0055
.76	.0502	.0035
.78	.0386	.0021
.80	.0289	.0012
.82	.0207	.0007
.84	.0144	.0003
.86	.0094	.0002
.88	.0057	.0001
.90	.0032	.0000
.92	.0015	.0000
.94	.0006	.0000
.96	.0002	.0000
.98	.0000	.0000
1.00	.0000	.0000

(a)

(b)

(c)
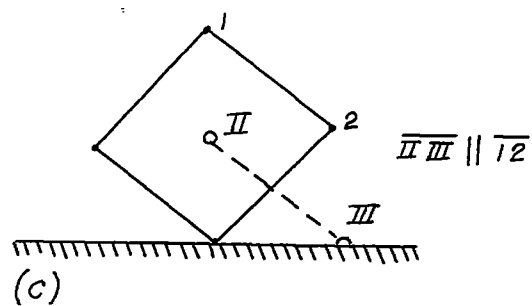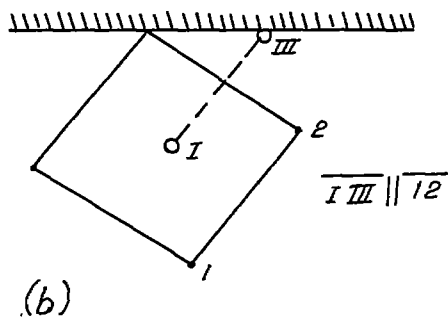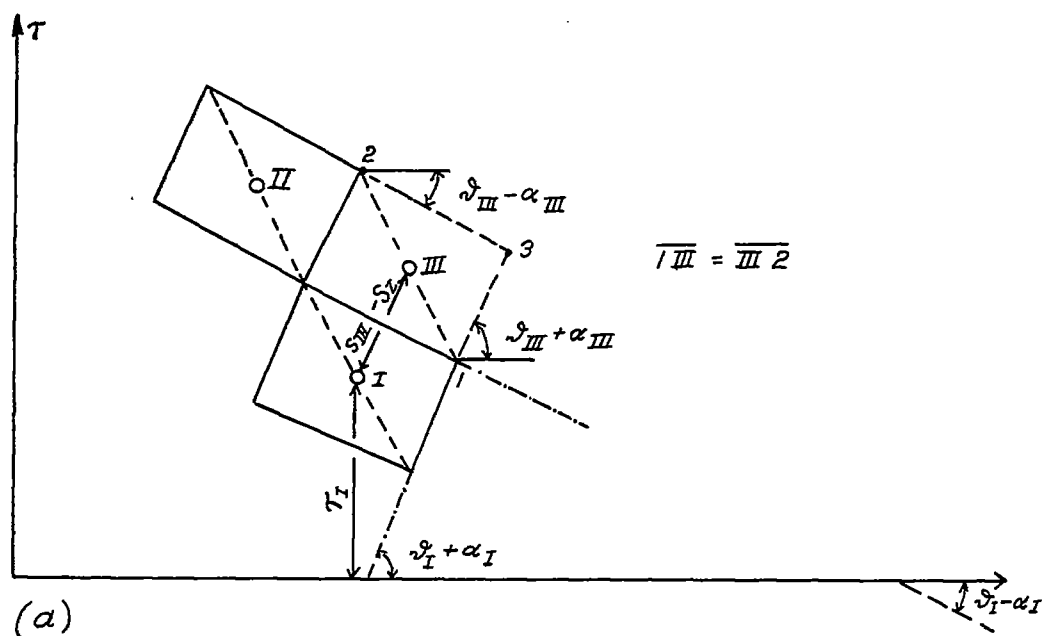
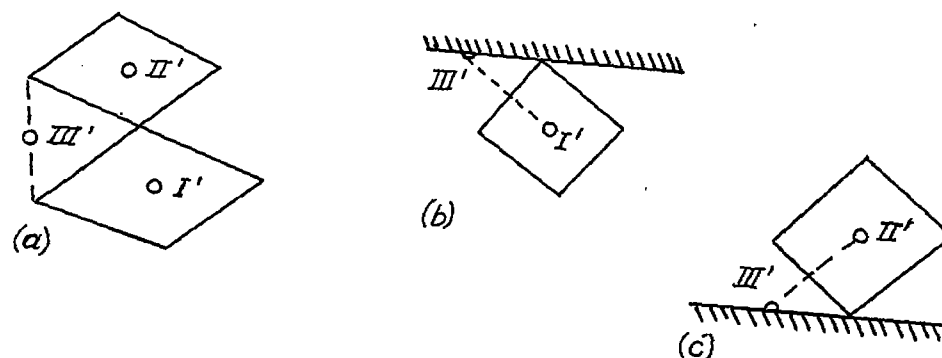ϑ_{III} prescribed by boundary condition
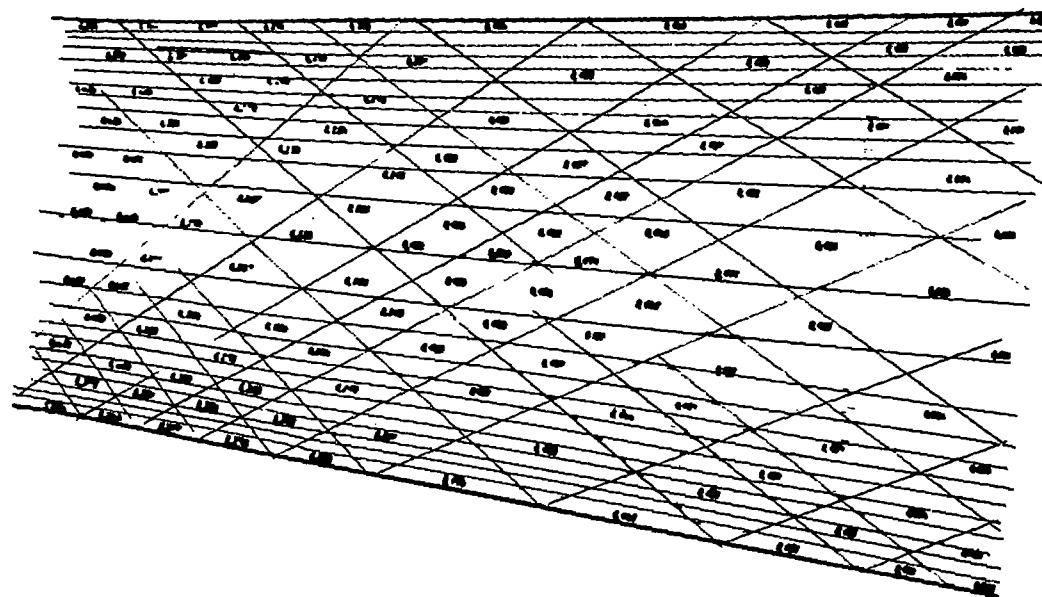
Figure I

Figure 2.

Figure 3. *

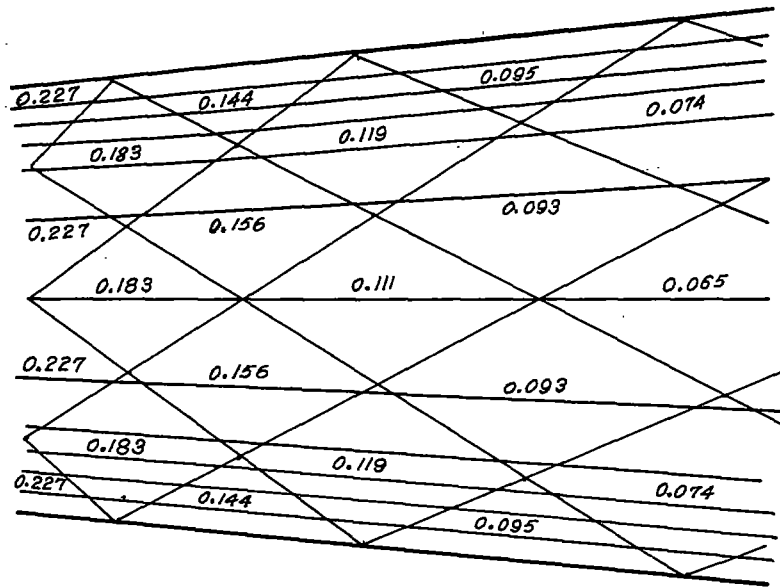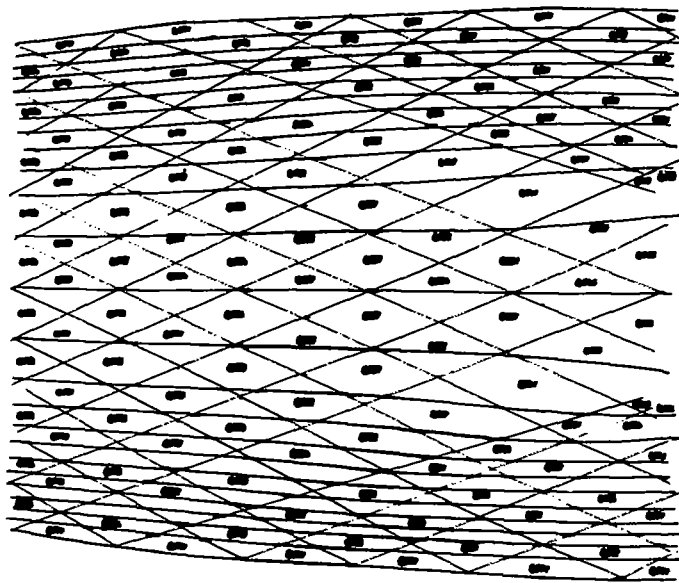*The values in this figure were illegible in the only available copy of the original German report.
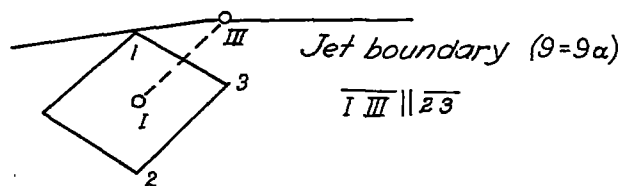
Figure 4.

Figure 5. *

Figure 6.

*The values in this figure were illegible in the only available copy of the
original German report.